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DISCRETE ORDINATE SOLUTION OF THE RADIATIVE TRANSFER EQUATION IN THE "POLARIZATION NORMAL WAVE REPRESENTATION"

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ABSTRACT

The transfer equations for normal waves in finite, inhomogeneous and plane-parallel magnetoactive media are solved using the discrete ordinate method. The physical process of absorption, emission, and multiple scattering are accounted for, and the medium may be forced both at the top and bottom boundary by anisotropic radiation as well as by internal anisotropic sources.

The computational procedure is numerically stable for arbitrarily large optical depths, and the computer time is independent of optical thickness.
Subject headings: polarization — radiative transfer

1. INTRODUCTION

To compute the spectrum and polarization of light emitted by many astrophysical sources it is necessary to solve the equation pertinent to radiation transport in a magnetoactive plasma. In general the radiative transfer equation must be solved for four Stokes parameters; however, for most astrophysical plasmas the concept of two normal waves—the ordinary and the extraordinary—applies (Gnedin & Pavlov 1974). The resulting radiative transfer equation has been studied by several authors under different assumptions, e.g., Ramaty (1969) and Meszáros, Nagel, & Ventura (1980); one of the more general solutions to the problem is given by Nagendra & Peraiah (1985). The main aim of this paper is to present a reliable and efficient solution of the equation pertinent to radiation transport in an absorbing, emitting and scattering, inhomogeneous, and magnetoactive plasma slab. In § 1.1 we present the appropriate radiative transfer equation and cast it in a form amenable to solution. Section 2 is devoted to the description of the solution of the resulting radiation transport equation in the discrete ordinate approximation. In § 3 we verify the solution method by comparing it with exact results. Finally in § 4 we briefly discuss possible applications and extensions of the theory.

1.1. *The Radiative Transfer Equation*

Assuming azimuthal symmetry and coherent (monoenergetic) scattering, the normal wave equations in slab geometry are given by (Gnedin & Pavlov 1974),

$$\mu \frac{d}{dz}\begin{bmatrix} I_1(z,\mu) \\ I_2(z,\mu) \end{bmatrix} = -\sum_k n_k(z)\begin{bmatrix} \sigma_{11}^{k,\mathrm{tot}} & 0 \\ 0 & \sigma_{22}^{k,\mathrm{tot}} \end{bmatrix}\begin{bmatrix} I_1(z,\mu) \\ I_2(z,\mu) \end{bmatrix} + \sum_k n_k(z)\int_{-1}^{1}\begin{bmatrix} \sigma_{11}^{k,\mathrm{sca}}(\mu;\mu') & \sigma_{12}^{k,\mathrm{sca}}(\mu;\mu') \\ \sigma_{21}^{k,\mathrm{sca}}(\mu;\mu') & \sigma_{22}^{k,\mathrm{sca}}(\mu;\mu') \end{bmatrix}\begin{bmatrix} I_1(z,\mu') \\ I_2(z,\mu') \end{bmatrix} d\mu' + \begin{bmatrix} Q_1(z,\mu) \\ Q_2(z,\mu) \end{bmatrix}. \tag{1}$$

Here $n_k(z)$ is the density of the kth species in the medium and $I_1(z, \mu)$ and $I_2(z, \mu)$ represent the extraordinary and ordinary wave, respectively, traveling in direction $\mu = \cos\theta$ (θ is the polar angle between the ray and the z-axis) at depth z. The mode exchange scattering coefficients are $\sigma_{12}^{k,\mathrm{sca}}(\mu;\mu')$ and $\sigma_{21}^{k,\mathrm{sca}}(\mu;\mu')$; and $\sigma_{11}^{k,\mathrm{sca}}(\mu;\mu')$ and $\sigma_{22}^{k,\mathrm{sca}}(\mu;\mu')$ are the mode conserving scattering coefficients. $Q_1(z,\mu)$ and $Q_2(z,\mu)$ represent general anisotropic sources embedded in the medium. For thermal radiation $Q_a(z,\mu) = \sigma_a^{\mathrm{abs}}B(T)/2$, where $B(T)$ is the Planck function at the local temperature T. The total cross sections are defined by

$$\sigma_{aa}^{\mathrm{tot}} = \int_{4\pi}\sigma_{aa}^{\mathrm{sca}}(\Theta)d\Omega + \int_{4\pi}\sigma_{aa}^{\mathrm{abs}}(\Theta)d\Omega\,, \qquad a = 1, 2\,. \tag{2}$$

To solve equation (1), we start by writing it on a form amenable to solution. For convenience we define an optical thickness

$$d\tau = -\sum_k n_k(z)\sigma_{11}^{k,\mathrm{tot}}\,dz \tag{3}$$

and phase matrix elements

$$\frac{1}{2}P_{ab}(z,\mu;\mu') = \frac{\sum_k n_k(z)\sigma_{ab}^{k,\mathrm{sca}}(\mu;\mu')}{\sum_k n_k(z)\sigma_{ab}^{k,\mathrm{sca}}}\,, \qquad a, b = 1, 2\,. \tag{4}$$

We also define a single scattering albedo

$$\omega_{11}(z) = \frac{\sum_k n_k(z)\sigma_{11}^{k,\mathrm{sca}}}{\sum_k n_k(z)\sigma_{11}^{k,\mathrm{tot}}}\,, \tag{5}$$

mode conversion factors for mode 1 to mode 2 scattering

$$f_{12}(z) = \frac{\sum_k n_k(z)\sigma_{12}^{k,\mathrm{sca}}}{\sum_k n_k(z)\sigma_{11}^{k,\mathrm{sca}}}, \tag{6}$$

and for mode 2 to mode 1 scattering

$$f_{21}(z) = \frac{\sum_k n_k(z)\sigma_{21}^{k,\mathrm{sca}}}{\sum_k n_k(z)\sigma_{11}^{k,\mathrm{sca}}}. \tag{7}$$

Furthermore we define a relative "scattering" depth between mode 2 and 1

$$f_{22}(z) = \frac{\sum_k n_k(z)\sigma_{22}^{k,\mathrm{sca}}}{\sum_k n_k(z)\sigma_{11}^{k,\mathrm{sca}}}, \tag{8}$$

and a relative "exctinction" depth between mode 2 and 1

$$f_{22}^{\mathrm{tot}}(z) = \frac{\sum_k n_k(z)\sigma_{22}^{k,\mathrm{tot}}}{\sum_k n_k(z)\sigma_{11}^{k,\mathrm{tot}}}. \tag{9}$$

In general plasmas are highly inhomogeneous, and the typical situation is the existence of smooth and very large changes in the properties of the medium. Hence, when solving the radiative transfer equation, the optical properties should be allowed to vary throughout the medium. The optical properties of the plasma slab under consideration are described by the single scattering albedo $\omega_{11}(\tau)$, the different mode factors $f_{ab}(\tau)$ and the phase matrix elements $P_{ab}(\tau, \mu; \mu')$ which are all functions of τ in an inhomogeneous medium. To allow for this τ-dependence the medium is assumed to consist of L adjacent homogeneous layers. In each layer the single scattering albedo, the mode factors and the phase matrix elements are taken to be constant, but they are allowed to vary from layer to layer. The source term is approximated by an exponential-linear function in τ in each layer; see § 2.2. For each homogeneous layer $\tau_{p-1} \le \tau \le \tau_p$, $p = 1, \ldots, L$ we may thus write equation (1) as

$$\mu \frac{d}{d\tau}\begin{bmatrix} I_1(\tau,\mu) \\ I_2(\tau,\mu)\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & f_{22}^{\mathrm{tot}}\end{bmatrix}\begin{bmatrix} I_1(\tau,\mu) \\ I_2(\tau,\mu)\end{bmatrix} - \frac{\omega_{11}}{2}\int_{-1}^{1}\begin{bmatrix} P_{11}(\mu;\mu') & f_{12}P_{12}(\mu;\mu') \\ f_{21}P_{21}(\mu;\mu') & f_{22}P_{22}(\mu;\mu')\end{bmatrix}\begin{bmatrix} I_1(\tau,\mu') \\ I_2(\tau,\mu')\end{bmatrix}d\mu' - \begin{bmatrix}\tilde{Q}_1(\tau,\mu) \\ \tilde{Q}_2(\tau,\mu)\end{bmatrix}, \tag{10}$$

where

$$\tilde{Q}_a(z,\mu) = \frac{Q_a(z,\mu)}{\sum_k n_k(z)\sigma_{11}^{k,\mathrm{tot}}}. \tag{11}$$

2. DISCRETE ORDINATE SOLUTION

To solve equation (10) we use the discrete ordinate method as developed by Chandrasekhar (1960) and Stamnes et al. (1988). First we discuss the solution of the homogeneous equation; next, we find a particular solution to the inhomogeneous equation.

2.1. *Homogeneous Solution*

Replacing the integral in equation (10) by a sum using Gaussian quadrature we get (homogeneous equation only)

$$\mu_i \frac{d}{d\tau}\begin{bmatrix} I_1(\tau,\mu_i) \\ I_2(\tau,\mu_i)\end{bmatrix} = \sum_{\substack{j=-N \\ j\neq 0}}^{N}\begin{bmatrix}\delta_{ij} - a_j D_{11}(\mu_i;\mu_j) & -a_j D_{12}(\mu_i;\mu_j) \\ -a_j D_{21}(\mu_i;\mu_j) & \delta_{ij} f_{22}^{\mathrm{tot}} - a_j D_{22}(\mu_i;\mu_j)\end{bmatrix}\begin{bmatrix} I_1(\tau,\mu_j) \\ I_2(\tau,\mu_j\end{bmatrix}, \qquad i = \pm 1, \pm 2, \ldots, \pm N, \tag{12}$$

where the μ_j's and the a_j's are the quadrature points and weights, respectively, and ($f_{11} = 1$)

$$D_{ab}(\mu_i;\mu_j) = \tfrac{1}{2} f_{ab}\,\omega_{11} P_{ab}(\mu_i;\mu_j). \tag{13}$$

We note that the merit of the discrete ordinate approximation is to replace two coupled linear integrodifferential equations (which have no known analytic solutions) with a system of $4N$ coupled linear differential equations for which analytical solutions (with numerical coefficients) exists.

Assuming that the phase matrix elements satisfy the following symmetry relations valid for scattering by a central force field

$$\begin{aligned} P_{ab}(\mu_i;\mu_j) &= P_{ab}(-\mu_i;-\mu_j) \\ P_{ab}(-\mu_i;\mu_j) &= P_{ab}(\mu_i;-\mu_j), \end{aligned} \tag{14}$$

we may write equation (12) in matrix form as

$$\frac{d}{d\tau}\begin{bmatrix} I_1^+(\tau) \\ I_1^-(\tau) \\ I_2^+(\tau) \\ I_2^-(\tau)\end{bmatrix} = \begin{bmatrix}\alpha & \beta & \zeta & \eta \\ -\beta & -\alpha & -\eta & -\zeta \\ \iota & \kappa & \gamma & \lambda \\ -\kappa & -\iota & -\lambda & -\gamma\end{bmatrix}\begin{bmatrix} I_1^+(\tau) \\ I_1^-(\tau) \\ I_2^+(\tau) \\ I_2^-(\tau)\end{bmatrix}, \tag{15}$$

where

$$I^{\pm}_{1,2}(\tau) = I_{1,2}(\tau, \pm\mu_i), \qquad i = 1, \ldots, N \tag{16}$$

and the $N \times N$ matrices $\alpha, \beta, \gamma, \lambda, \zeta, \eta, \iota$, and κ are

$$\alpha_{ij} = \frac{1}{\mu_i}[\delta_{ij} - a_j D_{11}(\mu_i; \mu_j)] = \frac{1}{\mu_i}[\delta_{ij} - a_j D_{11}(-\mu_i; -\mu_j)] \tag{17}$$

$$\beta_{ij} = \frac{1}{\mu_i}[-a_j D_{11}(\mu_i; -\mu_j)] = \frac{1}{\mu_i}[-a_j D_{11}(-\mu_i; \mu_j)] \tag{18}$$

$$\gamma_{ij} = \frac{1}{\mu_i}[\delta_{ij} f_{22}^{\text{tot}} - a_j D_{22}(\mu_i; u_j)] = \frac{1}{\mu_i}[\delta_{ij} f_{22}^{\text{tot}} - a_j D_{22}(-\mu_i; -\mu_j)] \tag{19}$$

$$\lambda_{ij} = \frac{1}{\mu_i}[-a_j D_{22}(\mu_i; -\mu_j)] = \frac{1}{\mu_i}[-a_j D_{22}(-\mu_i; \mu_j)] \tag{20}$$

$$\zeta_{ij} = \frac{1}{\mu_i}[-a_j D_{12}(\mu_i; \mu_j)] = \frac{1}{\mu_i}[-a_j D_{12}(-\mu_i; -\mu_j)] \tag{21}$$

$$\eta_{ij} = \frac{1}{\mu_i}[-a_j D_{12}(\mu_i; -\mu_j)] = \frac{1}{\mu_i}[-a_j D_{12}(-\mu_i; u_j)] \tag{22}$$

$$\iota_{ij} = \frac{1}{\mu_i}[-a_j D_{21}(\mu_i; \mu_j)] = \frac{1}{\mu_i}[-a_j D_{21}(-\mu_i; -\mu_j)] \tag{23}$$

$$\kappa_{ij} = \frac{1}{\mu_i}[-a_j D_{21}(\mu_i; -\mu_j)] = \frac{1}{\mu_i}[-a_j D_{21}(-\mu_i; \mu_j)] . \tag{24}$$

We have used the property $\mu_{-i} = -\mu_i$ which applies for the Gaussian quadrature rule adopted here.

In the case of no scattering, $\omega_{11} = 0$, all the off-diagonal elements in the $4N \times 4N$ matrix in equation (15) become zero and the transfer equations for the different components of $I_a^{\pm}$ decouple. This case of absorption and emission (source term included) has been studied by, e.g., Ramaty (1969). Furthermore, in the case of no mode conversion, i.e., $f_{12} = f_{21} = 0$, the equations for I_1 and I_2 separates and for each component we are left with a scalar integrodifferential equation which is readily solved using for example the discrete ordinate algorithm due to Stamnes et al. (1988).

In general $\omega_{11}, f_{12}, f_{21}$, and f_{22} may all be non zero. We then seek solutions to equation (15) of the form

$$I_1^{\pm}(\tau) = g^{\pm} e^{-k\tau} \tag{25}$$

$$I_2^{\pm}(\tau) = h^{\pm} e^{-k\tau} . \tag{26}$$

Insertion of equations (25)–(26) in equation (15) gives

$$\begin{bmatrix} -\alpha & -\beta & -\zeta & -\eta \\ \beta & \alpha & \eta & \zeta \\ -\iota & -\kappa & -\gamma & -\lambda \\ \kappa & \iota & \lambda & \gamma \end{bmatrix} \begin{bmatrix} g^+ \\ g^- \\ h^+ \\ h^- \end{bmatrix} = k \begin{bmatrix} g^+ \\ g^- \\ h^+ \\ h^- \end{bmatrix} \tag{27}$$

which is a standard eigenvalue problem of order $4N \times 4N$ determining the eigenvalues k and the eigenvectors $g^{\pm}$ and $h^{\pm}$. Because of the special structure of the matrix in equation (27) the eigenvalues occur in pairs ($\pm k$) and the order of the eigenvalue problem in equation (27) may be reduced by a factor 2 by generalizing a procedure introduced by Stamnes & Swanson (1981) for the scalar version of equation (1). First we rewrite equation (27) as

$$\alpha g^+ + \beta g^- + \zeta h^+ + \eta h^- = -k g^+ \tag{28}$$

$$\beta g^+ + \alpha g^- + \eta h^+ + \zeta h^- = \quad k g^- \tag{29}$$

$$\iota g^+ + \kappa g^- + \gamma h^+ + \lambda h^- = -k h^+ \tag{30}$$

$$\kappa g^+ + \iota g^- + \lambda h^+ + \gamma h^- = \quad k h^- . \tag{31}$$

By adding and subtracting the first two equations and the last two equations, we get

$$(\alpha + \beta)(g^+ + g^-) + (\zeta + \eta)(h^+ + h^-) = -k(g^+ - g^-) \tag{32}$$

$$(\alpha - \beta)(g^+ - g^-) + (\zeta - \eta)(h^+ - h^-) = -k(g^+ + g^-) \tag{33}$$

$$(\iota + \kappa)(g^+ + g^-) + (\gamma + \lambda)(h^+ + h^-) = -k(h^+ - h^-) \tag{34}$$

$$(\iota - \kappa)(g^+ - g^-) + (\gamma - \lambda)(h^+ - h^-) = -k(h^+ + h^-) . \tag{35}$$

Eliminating $(g^+ - g^-)$ and $(h^+ - h^-)$ in equations (32)–(35) gives the following eigenvalue problem of order $2N \times 2N$:

$$\begin{bmatrix} (\alpha-\beta)(\alpha+\beta)+(\zeta-\eta)(\iota+\kappa) & (\alpha-\beta)(\zeta+\eta)+(\zeta-\eta)(\gamma+\lambda) \\ (\iota-\kappa)(\alpha+\beta)+(\gamma-\lambda)(\iota+\kappa) & (\iota-\kappa)(\zeta+\eta)+(\gamma-\lambda)(\gamma+\lambda) \end{bmatrix} \begin{bmatrix} g^+ + g^- \\ h^+ + h^- \end{bmatrix} = k^2 \begin{bmatrix} g^+ + g^- \\ h^+ + h^- \end{bmatrix}, \tag{36}$$

which yields $2N$ values for k^2 and thereby determines the $4N$ eigenvalues $\pm k$ and the $2N$ eigenvectors $(g^+ + g^-)$ and $(h^+ + h^-)$. The eigenvectors $(g^+ - g^-)$ and $(h^+ - h^-)$ are determined from equations (32) and (34), respectively. The homogeneous solution may then be written as

$$I_1(\tau, \mu_i) = \sum_{\substack{l=-2N \\ l\neq 0}}^{2N} C_l g_l(\mu_i) e^{-k_l \tau} \tag{37}$$

$$I_2(\tau, \mu_i) = \sum_{\substack{l=-2N \\ l\neq 0}}^{2N} C_l h_l(\mu_i) e^{-k_l \tau}, \qquad i = \pm 1, \ldots, \pm N \tag{38}$$

where C_l are $4N$ constants of integration to be determined from the boundary conditions.

We note that in the 2 stream ($N = 1$) approximation, α, β, γ, λ, ζ, η, ι, and κ given in equations (17)–(24) are scalars; thus in this case the matrix on the left-hand side of equation (36) is a 2×2 matrix, and we may readily solve the eigenvalue problem.

2.2. *Inhomogeneous Solution*

As mentioned above the embedded source for thermal radiation $Q_a(\tau, \mu) = \sigma_a^{\rm abs} B(T)/2$. The behavior of the Planck function depends upon the frequency and temperature regions of interest. Integrated over the entire spectrum, the Planck function becomes

$$B(T) = \sigma T^4, \tag{39}$$

where σ is the Stefan-Boltzmann constant. In the Wien limit ($h\nu \gg kT$; h is the Planck constant and k is the Boltzmann constant) the Planck function at frequency ν is

$$B_\nu(T) = \frac{2h\nu^3}{c^2} e^{-h\nu/kT} \tag{40}$$

and in the Rayleigh-Jeans limit ($h\nu \ll kT$)

$$B_\nu(T) = \frac{2v^2}{c^2} kT. \tag{41}$$

Thus depending on the problem at hand, the source function may vary slowly or rapidly as a function of temperature, or depth z in the layer since $T = T(z)$.

To allow for internal source functions that may vary both rapidly and slowly throughout the layer, or with optical depth, we approximate the internal source function in each layer by an exponential-linear function (Kylling & Stamnes 1991)

$$\begin{bmatrix} \tilde{Q}_1(\tau, \mu) \\ \tilde{Q}_2(\tau, \mu) \end{bmatrix} = e^{-\rho\tau} \begin{bmatrix} X_1^0(\mu) + X_1^1(\mu)\tau \\ X_2^0(\mu) + X_2^1(\mu)\tau \end{bmatrix}, \tag{42}$$

where ρ, X_1^0, X_2^0, X_1^1, and X_2^1 are constants found by fitting the actual source to equation (42), as explained below. Seeking a particular solution of the form

$$\begin{bmatrix} I_1(\tau, \mu) \\ I_2(\tau, \mu) \end{bmatrix} = e^{-\rho\tau} \begin{bmatrix} Y_1^0(\mu) + Y_1^1(\mu)\tau \\ Y_2^0(\mu) + Y_2^1(\mu)\tau \end{bmatrix}, \tag{43}$$

we get after inserting equation (43) into equation (10) and equating coefficients of like powers of τ

$$\sum_{\substack{j=-N \\ j\neq 0}}^{N} \begin{bmatrix} \delta_{ij}(1+\rho\mu_i) - a_j D_{11}(\mu_i; \mu_j) & -a_j D_{12}(\mu_i; \mu_j) \\ -a_j D_{21}(\mu_i; \mu_j) & \delta_{ij}(f_{22}^{\rm tot} + \rho\mu_i) - a_j D_{22}(\mu_i; \mu_j) \end{bmatrix} \begin{bmatrix} Y_1^1(\mu_j) \\ Y_2^1(\mu_j) \end{bmatrix} = \begin{bmatrix} X_1^1(\mu_i) \\ X_2^1(\mu_i) \end{bmatrix} \tag{44}$$

$$\sum_{\substack{j=-N \\ j\neq 0}}^{N} \begin{bmatrix} \delta_{ij}(1+\rho\mu_i) - a_j D_{11}(\mu_i; \mu_j) & -a_j D_{12}(\mu_i; \mu_j) \\ -a_j D_{21}(\mu_i; \mu_j) & \delta_{ij}(f_{22}^{\rm tot} + \rho\mu_i) - a_j D_{22}(\mu_i; \mu_j) \end{bmatrix} \begin{bmatrix} Y_1^0(\mu_j) \\ Y_2^0(\mu_j) \end{bmatrix} = \begin{bmatrix} X_1^0(\mu_i) \\ X_2^0(\mu_i) \end{bmatrix} + \mu_i \begin{bmatrix} Y_1^1(\mu_i) \\ Y_2^1(\mu_i) \end{bmatrix}, \tag{45}$$

which are linear algebraic equations of the form $Ax = b$ determining the $Y_a^0(\mu_i)$ and $Y_a^1(\mu_i)$ coefficients, $a = 1, 2$. We may thus write the full solution to equation (1) for layer p

$$I_{1,p}(\tau, \mu_i) = \sum_{\substack{l=-2N \\ l\neq 0}}^{2N} C_{l,p} g_{l,p}(\mu_i) e^{-k_{l,p}\tau} + e^{-\rho_p \tau}[Y_{1,p}^0(\mu_i) + Y_{1,p}^1(\mu_i)\tau] \tag{46}$$

$$I_{2,p}(\tau, \mu_i) = \sum_{\substack{l=-2N \\ l\neq}}^{2N} C_{l,p} h_{l,p}(\mu_i) e^{-k_{l,p}\tau} + e^{-\rho_p \tau}[Y_{2,p}^0(\mu_i) + Y_{2,p}^1(\mu_i)\tau], \qquad i = \pm 1, \ldots, \pm N. \tag{47}$$

It should be noticed that equations (46)–(47) allow us to compute the intensity fields at any optical depth in the medium.

2.2.1 *The ρ, X_1^0, X_2^0, X_1^1, and X_2^1 Coefficients*

To find the five coefficients, ρ, X_1^0, X_2^0, X_1^1, and X_2^1, five equations are required. We assume that the internal sources are known at the layer boundaries τ_0 and τ_2, $\tau_2 > \tau_0$, and that $\tilde{Q}_1$ is known at $\tau_1 = (\tau_0 + \tau_2)/2$.

$$\tilde{Q}_1(\tau_0, \mu) = e^{-\rho\tau_0}[X_1^0(\mu) + X_1^1(\mu)\tau_0]$$
$$\tilde{Q}_1(\tau_1, \mu) = e^{-\rho\tau_1}[X_1^0(\mu) + X_1^1(\mu)\tau_1]$$
$$\tilde{Q}_1(\tau_2, \mu) = e^{-\rho\tau_2}[X_1^0(\mu) + X_1^1(\mu)\tau_2]$$
$$\tilde{Q}_2(\tau_0, \mu) = e^{-\rho\tau_0}[X_2^0(\mu) + X_2^1(\mu)\tau_0]$$
$$\tilde{Q}_2(\tau_2, \mu) = e^{-\rho\tau_2}[X_2^0(\mu) + X_2^1(\mu)\tau_2] .$$

We force ρ to be angle independent by solving the equations for one fixed angle, e.g., μ_1. Solving the five equations above then gives

$$\rho = \frac{2}{\tau_2 - \tau_0} \ln \left\{ \frac{\tilde{Q}_1(\tau_1, \mu_1)}{\tilde{Q}_1(\tau_2, \mu_1)} \pm \sqrt{\left(\frac{\tilde{Q}_1(\tau_1, \mu_1)}{\tilde{Q}_1(\tau_2, \mu_1)}\right)^2 - \frac{\tilde{Q}_1(\tau_0, \mu_1)}{\tilde{Q}_1(\tau_2, \mu_1)}} \right\} \tag{48}$$

$$X_1^1(\mu_i) = \frac{\tilde{Q}_1(\tau_2, \mu_i)e^{\rho\tau_2} - \tilde{Q}_1(\tau_0, \mu_i)e^{\rho\tau_0}}{\tau_2 - \tau_0} \tag{49}$$

$$X_1^0(\mu_i) = \tilde{Q}_1(\tau_0, \mu_i)e^{\rho\tau_0} - X_1^1(\mu_i)\tau_0 \tag{50}$$

$$X_2^1(\mu_i) = \frac{\tilde{Q}_2(\tau_2, \mu_i)e^{\rho\tau_2} - \tilde{Q}_2(\tau_0, \mu_i)e^{\rho\tau_0}}{\tau_2 - \tau_0} \tag{51}$$

$$X_2^0(\mu_i) = \tilde{Q}_2(\tau_0, \mu_i)e^{\rho\tau_0} - X_2^1(\mu_i)\tau_0 , \qquad i = \pm 1, \ldots, \pm N . \tag{52}$$

Care must be taken in evaluating ρ in order to avoid numerical overflow in equations (49)–(52); see Kylling & Stamnes (1991). For $\tilde{Q}_1(\tau_2, \mu_i) > \tilde{Q}_1(\tau_1, \mu_i) > \tilde{Q}_1(\tau_0, \mu_i)$ we use the (+) solution for ρ and the (−) solution if $\tilde{Q}_1(\tau_2, \mu_i) < \tilde{Q}_1(\tau_1, \mu_i) < \tilde{Q}_1(\tau_0, \mu_i)$.

2.3. *Scaling Transformations*

To avoid overflow in equations (46)–(47) and most importantly numerical ill-conditioning when solving for the constants of integration (eqs. [65]–[70] below, remember $k_{-l} = -k_l$), we utilize the scaling transformation discussed by Stamnes & Conklin (1984). (Here and below we let $k_{-l} = -k_l$, thus $k_l > 0$ for all l.)

$$C_{+l,p} = \tilde{C}_{+l,p} e^{k_{l,p}\tau_{p-1}} \tag{53}$$
$$C_{-l,p} = \tilde{C}_{-l,p} e^{-k_{l,p}\tau_p} . \tag{54}$$

Inserting equations (53)–(54) into equations (46)–(47) we get

$$I_{1,p}(\tau, \mu_i) = \sum_{l=1}^{2N} [\tilde{C}_{-l,p} g_{-l,p}(\mu_i) e^{-k_{l,p}(\tau_p - \tau)} + \tilde{C}_{+l,p} g_{+l,p}(\mu_i) e^{-k_{l,p}(\tau - \tau_{p-1})}] + R_{1,p}(\tau, \mu_i) \tag{55}$$

$$I_{2,p}(\tau, \mu_i) = \sum_{l=1}^{2N} [\tilde{C}_{-l,p} h_{-l,p}(\mu_i) e^{-k_{l,p}(\tau_p - \tau)} + \tilde{C}_{+l,p} h_{+l,p}(\mu_i) e^{-k_{l,p}(\tau - \tau_{p-1})}] + R_{2,p}(\tau, \mu_i) , \tag{56}$$

where

$$R_{1,p}(\tau, \mu_i) = e^{-\rho_p\tau}[Y_{1,p}^0(\mu_i) + Y_{1,p}^1(\mu_i)\tau] \tag{57}$$
$$R_{2,p}(\tau, \mu_i) = e^{-\rho_p\tau}[Y_{2,p}^0(\mu_i) + Y_{2,p}^1(\mu_i)\tau] \qquad i = \pm 1, \ldots, \pm N , \tag{58}$$

and all the exponentials in the homogeneous solution have negative arguments as they should to avoid numerical overflow. We may thus proceed to calculate the constants of integration.

2.4. *Boundary Conditions*

We allow the medium to be illuminated from above by an anisotropic source

$$I_{1,1}(0, -\mu_i) = I_1^\infty(-\mu_i) \tag{59}$$
$$I_{2,1}(0, -\mu_i) = I_2^\infty(-\mu_i) , \qquad i = 1, \ldots, N . \tag{60}$$

Across layer interfaces we require the intensities to be continuous:

$$I_{1,p}(\tau_p, \mu_i) = I_{1,p+1}(\tau_p, \mu_i) \tag{61}$$
$$I_{2,p}(\tau_p, \mu_i) = I_{2,p+1}(\tau_p, \mu_i) , \qquad i = \pm 1, \ldots, \pm N , \qquad p = 1, \ldots, L-1 . \tag{62}$$

Finally the medium may be forced by radiation incident at the bottom boundary

$$I_{1,L}(\tau_L, \mu_i) = I_1^b(\mu_i) \tag{63}$$

$$I_{2,L}(\tau_L, \mu_i) = I_2^b(\mu_i)\,, \qquad i = 1, \ldots, N\,. \tag{64}$$

Insertion of equations (55)–(56) into equations (59)–(64) yields

$$\sum_{l=1}^{2N} [\tilde{C}_{-l,1} g_{-l,1}(-\mu_i) e^{-k_{l,1}\tau_1} + \tilde{C}_{+l,1} g_{+l,1}(-\mu_i)] = I_1^{\infty}(-\mu_i) - R_{1,1}(0, -\mu_i) \tag{65}$$

$$\sum_{l=1}^{2N} [\tilde{C}_{-l,1} h_{-l,1}(-\mu_i) e^{-k_{l,1}\tau_1} + \tilde{C}_{+l,1} h_{+l,1}(-\mu_i)] = I_2^{\infty}(-\mu_i) - R_{2,1}(0, -\mu_i)\,, \qquad i = 1, \ldots, N \tag{66}$$

$$\sum_{l=1}^{2N} [\tilde{C}_{-l,p} g_{-l,p}(\mu_i) + \tilde{C}_{+l,p} g_{+l,p}(\mu_i) e^{-k_{l,p}(\tau_p - \tau_{p-1})} - \tilde{C}_{-l,p+1} g_{-l,p+1}(\mu_i) e^{-k_{l,p+1}(\tau_{p+1} - \tau_p)} - \tilde{C}_{+l,p+1} g_{+l,p+1}(\mu_i)]$$
$$= R_{1,p+1}(\tau_p, \mu_i) - R_{1,p}(\tau_p, \mu_i) \tag{67}$$

$$\sum_{l=1}^{2N} [\tilde{C}_{-l,p} h_{-l,p}(\mu_i) + \tilde{C}_{+l,p} h_{+l,p}(\mu_i) e^{-k_{l,p}(\tau_p - \tau_{p-1})} - \tilde{C}_{-l,p+1} h_{-l,p+1}(\mu_i) e^{-k_{l,p+1}(\tau_{p+1} - \tau_p)} - \tilde{C}_{+l,p+1} h_{+l,p+1}(\mu_i)]$$
$$= R_{2,p+1}(\tau_p, \mu_i) - R_{2,p}(\tau_p, \mu_i)\,, \qquad i = \pm 1, \ldots, \pm N \qquad p = 1, \ldots, L-1 \tag{68}$$

$$\sum_{l=1}^{2N} [\tilde{C}_{-l,L} g_{-l,L}(\mu_i) e^{-k_{l,L}(\tau_L - \tau_{L-1})} + \tilde{C}_{+l,L} g_{+l,L}(\mu_i)] = I_1^b(\mu_i) - R_{1,L}(\tau_L, \mu_i) \tag{69}$$

$$\sum_{l=1}^{2N} [\tilde{C}_{-l,L} h_{-l,L}(\mu_i) e^{-k_{l,L}(\tau_L - \tau_{L-1})} + \tilde{C}_{+l,L} h_{+l,L}(\mu_i)] = I_2^b(\mu_i) - R_{2,L}(\tau_L, \mu_i)\,, \qquad i = 1, \ldots, N \tag{70}$$

which is a $(4N \times L) \times (4N \times L)$ system of linear algebraic equations from which the $4N \times L$ unknown coefficients $\tilde{C}_{l,p}$, $l = \pm 1, \ldots, \pm 2N$, $p = 1, \ldots, L$ are determined. The coefficient matrix is a $12N - 1$ diagonal band matrix. This fact may be used to speed up the computation of the constants of integration. It should be noted that due to the use of the scaling transformation equations (53)–(54), the system of linear equations given by equations (65)–(70) is unconditionally well-conditioned for arbitrary total optical thickness and arbitrary individual layer optical depths.

3. VERIFICATION OF THE SOLUTION METHOD

Complex numerical schemes such as the one outlined above need to be extensively tested and checked for errors before they can be safely used. For some special cases analytic solutions are available; furthermore comparisons can be made with previous solutions to similar problems. Our solutions, equations (55)–(56), are the sum of the homogeneous and the inhomogeneous solutions. We first verify that our homogeneous solution is correct and next the inhomogeneous solution.

3.1. *Homogeneous Solution*

To verify the numerical scheme for the homogeneous solution we repeat the calculations done by Chandrasekhar (1960, p. 234–249) for a conservative, semi-infinite electron scattering atmosphere. Since our model pertains to a finite atmosphere, we mimic the semi-infinite slab by setting the total optical depth equal to 100. For conservative scattering, $\omega_{11} = 1.$ and $f_{ab} = 1.$, the eigenvalue problem in equation (36) is degenerate, i.e., some eigenvalues are zero. However a value close to 1.0 will be adequate; we chose $\omega_{11} = 0.9999$. To make the analogy between equation (10) in this work and equation (227), Chandrasekhar (1960, p. 43) we make the identifications $I_1(\tau, \mu) = I_l(\tau, \mu)$ and $I_2(\tau, \mu) = I_r(\tau, \mu)$; furthermore; the phase matrix elements for an electron scattering atmosphere are given by

$$P_{11}(\mu; \mu') = \tfrac{3}{2}(1 - \mu^2)(1 - \mu'^2) + \tfrac{3}{4}\mu^2\mu'^2$$
$$P_{12}(\mu; \mu') = \tfrac{3}{4}\mu^2$$
$$P_{21}(\mu; \mu') = \tfrac{3}{4}\mu'^2$$
$$P_{22}(\mu; \mu') = \tfrac{3}{4}\,.$$

We use a 20 stream ($N = 10$) approximation and have isotropic radiation incident at the bottom of the slab. Figure 1 shows the degree of polarization as obtained by Chandrasekhar (*solid line*) and by the present calculation (*asterisks*). Good agreement is found between the two methods.

The atmosphere discussed by Chandrasekhar is homogeneous, therefore, one layer is sufficient in our model. Dividing the atmosphere into several identical layers with the same total thickness should of course produce the same results. Such a multilayer calculation provides a good test for checking the calculation of the constants of integration in equations (65)–(70) and was thus performed. The same result was obtained as in the single-layer calculation.

3.2. *Inhomogeneous Solution*

To test the inhomogeneous solution for an anisotropic source $Q(\tau, \mu)$ we set $f_{12} = f_{21} = 0$. Equation (10) then separates into two independent equations, each of the same form as the much studied scalar equation for the diffuse radiation field in a plane-parallel

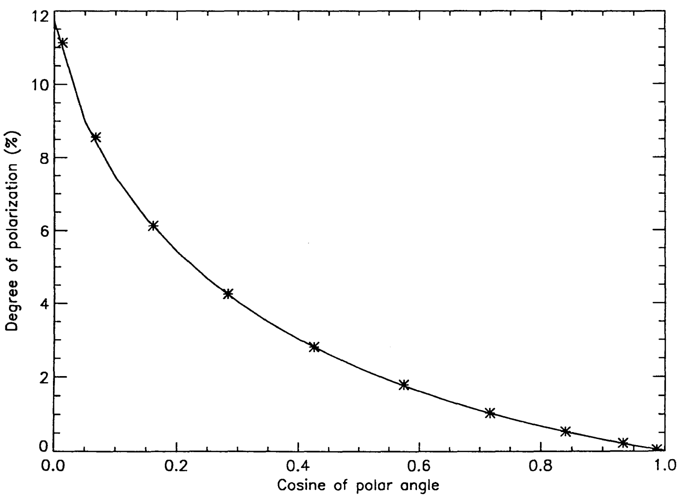

FIG. 1.—The degree of polarization $\delta = [I_l(\tau, \mu) - I_r(\tau, \mu)]/[I_l(\tau, \mu) + I_r(\tau, \mu)]$ as a function of the polar angle θ. The solid line is the result of Chandrasekhar; the asterisks represent the present method.

atmosphere. Using the attenuated direct beam as an example of an anisotropic internal pseudosource (Kylling & Stamnes 1991), equation (10) was solved for a variety of different phase matrix elements $P_{11}(\mu; \mu')$ and $P_{22}(\mu; \mu')$ and optical depths τ. The results were compared with solutions from the scalar equation for the diffuse radiation field, as solved by Stamnes et al. (1988). Excellent agreement was found.

4. DISCUSSION

The flexibility and generality of the above formalism makes it applicable to detailed studies of radiation from a variety of astrophysical sources, e.g., X-ray pulsars. The effects of inhomogeneties in the medium that the radiation traverses may readily be studied, and both the emergent and internal radiation fields are easily calculated.

It is noted that with the appropriate phase elements, the above solution method, combined with the solution of the scalar version of equation (10), constitute the complete solution for a layered Rayleigh-scattering atmosphere; see Chandrasekhar (1960) p. 249–256). Furthermore the equivalence between equation (1) and the mono-energetic one-dimensional transport equation pertinent to hydrogen/proton transport in Earth's upper atmosphere (Rees 1989) is noted.

The author believes that the above formalism may be extended to include azimuthal dependence and all four components of the Stokes vector. To include incoherent scattering, the multigroup procedure (Duderstadt & Martin 1979) may be used.

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REFERENCES

Chandrasekhar, S. 1960, Radiative Transfer (New York: Dover)
Duderstadt, J. J., & Martin, W. R. 1979, Transport Theory (New York: Wiley)
Gnedin, Yu., N., & Pavlov, G. G. 1974, Soviet Phys.—JETP, 38, 903
Kylling, A., & Stamnes, K. 1991, J. Comp. Phys., submitted
Meszáros, P., Nagel, W., & Ventura, J. 1980, ApJ, 238, 1066
Nagendra, K. N., & Peraiah, A. 1985, MNRAS, 214, 203
Ramaty, R. 1969, ApJ, 158, 753
Rees, M. H. 1989, Physics and Chemistry of the Upper Atmosphere (Cambridge: Cambridge Univ. Press)
Stamnes, K., & Conklin, P. 1984, J. Quant. Spectrosc. Radiat. Trans., 31, 273
Stamnes, K., Tsay, S.-C., Wiscombe, W., & Jayaweera, K. 1988, Appl. Optics, 27, 2502
Stamnes, K., & Swanson, R. A. 1981, J. Atmos. Sci., 38, 387